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Plane Jet in a Moving Medium

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It is well known that major difficulties in the viscous theory of jets occur from the impossibility of obtaining similar solutions when conditions are different from those of Schlichting.[2]

The purpose of this paper is to show how it is possible to obtain "nearly similar" solutions and to give a solution of the linearized equation.

Basic Equations and "Nearly Similar" Solutions

The equations governing the motion of an incompressible plane jet in a moving medium in the boundary-layer approximation are‡

$$u_x + v_y = 0 \tag{1}$$

$$uu_x + vu_y = \nu u_{yy} \tag{2}$$

Related boundary conditions are

$$\begin{aligned} u(0, y) &= u_i \quad \text{for} \quad y < h \\ u(0, y) &= u_e \quad \text{for} \quad y > h \end{aligned} \tag{3}$$

$$\begin{aligned} v(x,0) &= 0 \qquad u_y(x,0) = 0 \\ u(x, \infty) &= u_e \end{aligned} \tag{4}$$

where $2h$ is the height of the jet at $x = 0$, u_i its velocity, and u_e the external (constant) velocity.

If it is assumed[3] that the slit is infinitesimal, initial conditions [Eq. (3)] are replaced by

$$\frac{d}{dx}\int_{-\infty}^{\infty}(u^2 - uu_e)dy = 0 \tag{5}$$

(This equation was obtained by integrating Eq. (2) between $-\infty$ and $+\infty$, taking into account the continuity equation).

Put in Eqs. (1) and (2)

$$\bar{u} = u - u_e \qquad \bar{v} = v \tag{6}$$

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‡ These equations also hold good for compressible fluids as long as $\mu\rho$ can be assumed constant.

The continuity equation does not change, whereas the momentum equation becomes

$$\bar{u}\bar{u}_x + \bar{v}\bar{u}_y + u_e\bar{u}_x = \nu\bar{u}_{yy} \tag{7}$$

Now one assumes the stream function ψ to be a "nearly similar" function; i.e., one writes

$$\psi(x, y) = F(x)\sum_{i=0}^{\infty} f_i(\eta)m^i(x) \tag{8}$$

with

$$\eta = y/h(x) \tag{9}$$

Equations and boundary conditions make it possible to find the unknown functions $h(x)$, $m(x)$, and $F(x)$ to obtain

$$h = 3\nu^{1/2}x; \qquad m = 3u_e x^{1/3}; \qquad F(x) = \nu^{1/3}x^{1/3} \tag{10}$$

Note that the $f_0(\eta)$ in Eq. (10) is that of Schlichting's solution[3] for a jet in a medium at rest.

To obtain the velocity functions $f_i(\eta)$ of the expansion scheme (8), this expression is substituted in Eq. (7) and the $m^i(x)$ coefficients equated to zero.

Solutions of the Linearized Equation

As long as m is sufficiently small (i.e., $m^2 \ll m$), only the first term in the series expansion of Eq. (8) may be considered. The equation that determines the velocity function $f_1'(\eta)$ in Eq. (9) is then

$$f_1''' = -f_0'f_1' - 2f_0''f_1 - f_1''f_0 - \eta f_0'' - f_0' \tag{11}$$

subject to the boundary conditions

$$f_1(0) = 0 \qquad f_1''(0) = 0 \qquad f_1'(\infty) = 0 \tag{11$'$}$$

Moreover, to satisfy Eq. (5) up to terms of order m, it must be verified that

$$f_0(\infty) + 2\int_0^{\infty} f_0'f_1'\,d\eta = 0 \tag{11$''$}$$

Since it seems very difficult to solve Eq. (11) exactly, the following approximation is proposed:

Integrate Eq. (11) once:

$$f_1'' = -f_0'f_1' + 2f_0'\eta + f_0 - 2\int_0^{\eta} f_1f_0''\,d\eta \tag{12}$$

Now, as previously stated, condition (5) requires that

$$G(\eta) = f_0(\eta) - 2\int_0^{\eta} f_1f_0''\,d\eta \tag{13}$$

is zero when $\eta \to \infty$. Moreover, at $\eta = 0$ both function f_0 and $\int_0^{\eta} f_1f_0''\,d\eta$ are zero.

So, following an interative method, one assumes in Eq. (12), as first-approximation value of (13), $G(\eta) = 0$.

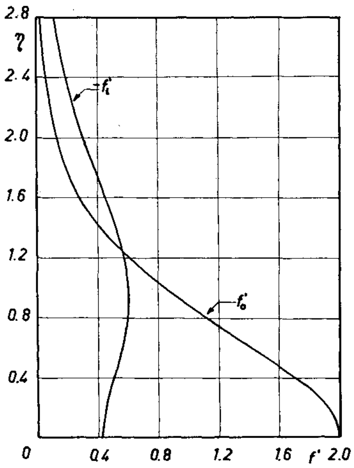

Fig. 1 Velocity functions

Equation (12) then can be solved easily to obtain, after imposing conditions (11)′,

$$f_1' = (C - \eta^2)/\cosh^2\eta \tag{14}$$

where C is a free constant. This constant makes it possible to satisfy condition (11)″ exactly.

The value of C so determined, which represents the u_1 component of the velocity at $\eta = 0$, is $C = -0.428$ and agrees with the numerical result, $C = 0{,}425$, of Ref. 1. Moreover, comparison of the forementioned solution with that of Ref. 3 shows that the agreement holds for the whole field, the difference being only of some units in the third decimal place. It therefore follows that a second iteration is not necessary. The velocity function (14) is shown in Fig. 1 together with the Schlichting solution for the jet in a medium at test.[3]

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Torsional Vibration of a Semi-Infinite Viscoelastic Circular Cylinder Due to Transient Torsional Couple

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This paper is concerned with the determination of displacement in a semi-infinite viscoelastic cylinder when a torque, exponentially decreasing with time, is applied on a prescribed region of the plane end.

Method of Solution

Let b be the radius of the circular cylinder, and let the torque be applied at the plane boundary $Z = 0$. The material of this cylinder is supposed to satisfy the stress-strain relation

$$\tau_{ij} = [\lambda + \lambda'(\partial/\partial t)]e_{kk}\delta_{ij} + [\mu + \mu'(\partial/\partial t)]e_{ij} \tag{1}$$

where τ_{ij} and e_{ij} are the stress and strain tensors, respectively, and

$$\delta_{ij} = 0 \qquad i \neq j$$
$$\delta_{ij} = 1 \qquad i = j$$

$\lambda, \lambda', \mu, \mu'$ being material constants.[1]

Choosing cylindrical coordinates with the Z axis along the axis of the cylinder, the components of displacement at any point of the cylinder are

$$u_r = 0 \qquad u_\theta = \vartheta(r,z,t) \qquad u_z = 0 \tag{2}$$

Thus the components of strain are

$$\begin{aligned} e_{rr} = e_{\theta\theta} = e_{zz} = 0 \qquad & e_{r\theta} = (\partial\vartheta/\partial r) - (\vartheta/r) \\ e_{\theta z} = \partial\vartheta/\partial z \qquad & e_{rz} = 0 \end{aligned} \tag{3}$$

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From (1), the stress components are

$$\begin{aligned} \tau_{rr} = \tau_{\theta\theta} = \tau_{zz} = 0 \qquad & \tau_{r\theta} = [\mu + \mu'(\partial/\partial t)]e_{r\theta} \\ \tau_{\theta z} = [\mu + \mu'(\partial/\partial t)]e_{\theta z} \qquad & \tau_{rz} = 0 \end{aligned} \tag{4}$$

The equations of motion take the form

$$\rho\frac{\partial^2\vartheta}{\partial t^2} = \left(\mu + \mu'\frac{\partial}{\partial t}\right)\left(\frac{\partial^2\vartheta}{\partial r^2} + \frac{1}{r}\frac{\partial\vartheta}{\partial r} - \frac{\vartheta}{r^2} + \frac{\partial^2\vartheta}{\partial z^2}\right) \tag{5}$$

ρ being the density of the material, with the boundary conditions

$$\begin{aligned} \tau_{\theta z} &= \left[\left(\mu + \mu'\frac{\partial}{\partial t}\right)\frac{\partial\vartheta}{\partial z}\right]_{z=0} = Pre^{-\Omega t} \qquad t \geqslant 0, \Omega > 0 \text{ for } 0 \leqslant r \leqslant a \\ &= 0 \qquad \text{for } a < r \leqslant b \text{ where } a < b \qquad (6) \\ \tau_{r\theta} &= \left[\left(\mu + \mu'\frac{\partial}{\partial t}\right)\left(\frac{\partial\vartheta}{\partial r} - \frac{\vartheta}{r}\right)\right]_{r=b} = 0 \qquad t \geqslant 0 \\ \vartheta &= 0 \qquad \text{at } z = \infty \end{aligned}$$

Assuming

$$\vartheta = A_nJ_1(K_nr)\varphi_n(z,t) \tag{7}$$

where K_nb is the nth root of $J_2(Kb) = 0$, Eq. (5) reduces to

$$\rho\frac{\partial^2\varphi_n}{\partial t^2} = \left(\mu + \mu'\frac{\partial}{\partial t}\right)\left[\frac{\partial^2\varphi_n}{\partial z^2} - K_n^2\varphi_n(z,t)\right] \tag{8}$$

To solve (8), take

$$\varphi_n(z,t) = f(z)e^{-\Omega t} \tag{9}$$

From (8) and (9), one obtains

$$(d^2f/dz^2) - m^2f = 0 \tag{10}$$

where $m^2 = (\rho\Omega^2 - K_n^2\Omega\mu' + K_n^2\mu)/(\mu - \mu'\Omega)$. It is assumed that $\mu > \mu'\Omega$.

Thus $f(z) = C_1e^{mz} + C_2e^{-mz}$. The last condition of (6) gives $C_1 = 0$, and $f(z)$ becomes

$$f(z) = C_2e^{-mz} \tag{11}$$

Thus one writes

$$\vartheta = \sum_{n=1}^{\infty} B_n\,J_1(K_nr)e^{-mz}e^{-\Omega t} \tag{12}$$

From the first condition of (6) and from (12), one obtains

$$\frac{Pr}{\mu - \mu'\Omega} = \sum_{n=1}^{\infty} C_nJ_1(K_nr) \tag{13}$$

where $C_n = -B_nm$. C_n is given by the relation[2]

$$C_n\{(K_n^2a^2 - 1)J_1^2(K_na) + a^2K_n^2J_1'^2(K_na)\} = 2K_n^2\int_0^a \frac{Pr^2}{\mu - \mu'\Omega}J_1(K_nr)dr = \frac{2K_n^2P}{(\mu - \mu'\Omega)}\cdot\frac{a^2}{K_n}J_2(K_na)$$

Therefore

$$C_n = \frac{2K_nPa^2J_2(K_na)}{(\mu - \mu'\Omega)[(K_n^2a^2 - 1)J_1^2(K_na) + a^2K_n^2J_1'^2(K_na)]}$$

From (13)

$$B_n = -C_n/m$$
$$= \frac{-2k_nPb^2J_2(K_na)}{(\mu - \mu'\Omega)m[(K_n^2a^2 - 1)J_1^2(K_na) + a^2K_n^2J_1'^2(K_na)]} \tag{14}$$